About Knop's action of the Weyl group on the set of orbits of a spherical subgroup in the flag manifold

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1 Introduction

Let G be a complex connected reductive algebraic group. Let $\mathcal{B}$ denote the flag variety of G. Let H be an algebraic subgroup of G which has a finite number of orbits in $\mathcal{B}$; H is said to be *spherical*. We denote by $\mathbf{H}(\mathcal{B})$ the set of the H-orbits in $\mathcal{B}$. The closures of these orbits are of importance in representation theory (see [Wol93]). Moreover, the elements of $\mathbf{H}(\mathcal{B})$, viewed as orbits of a Borel subgroup of G in G/H play an important role in the geometry and topology of the G-equivariant embeddings X of G/H.

In [Kno95], F. Knop introduced an action of a monoid (constructed from the Weyl group of G) on $\mathbf{H}(\mathcal{B})$. This action is called "weak order" and studied by M. Brion in [Bri01]. But, the most spectacular combinatoric structure of the set $\mathbf{H}(\mathcal{B})$ was discovered by F. Knop in [Kno95]: he defined an action of the Weyl group W of G on $\mathbf{H}(\mathcal{B})$. Actually, the results of F. Knop are stated in a more general context. The proof of the existence of this action is very indirect and sophisticated. The aim of this note is to construct natural invariants separating the W-orbits. Note that our methods are elementary.

Let us fix a maximal torus T^H of H. Denote by W_H the Weyl group of T^H. Let T be a maximal torus of G containing T^H and let W denote the Weyl group of T.

Let $V \in \mathbf{H}(\mathcal{B})$. Let x be a point of V whose the orbit by T^H is of minimal dimension. Denote by S the identity component of the stabilizer of x in T^H. The group W_H acts naturally on the set of subtori of T^H. The W_H-orbit of S is called *the type of* V. It is shown in Section 3 that the type of V only depends on V and not on x.

The main result of this note is the following

Theorem *Two elements of* $\mathbf{H}(\mathcal{B})$ *are in the same* W*-orbit for Knop's action if and only if they have the same type.*

In Section 2, we recall some useful definitions about a graph with vertices the elements of $\mathbf{H}(\mathcal{B})$, Knop's action of W on $\mathbf{H}(\mathcal{B})$ and some classical invariants associated to the elements of $\mathbf{H}(\mathcal{B})$. In Section 3, we show that the definition of the type of an orbit of H is consistent. After, we study the fixed points of subtori of H in the elements of $\mathbf{H}(\mathcal{B})$. In Section 5, we state and prove our main results. In the following one, we give some consequences of our results and our proofs.

2 Definitions and notation

2.1 — Let us fix some general notation. If Γ denotes a linear algebraic group, we denote by Γ° its identity component. If Γ acts on an algebraic variety X and x belongs to X, we denote by Γ_x the stabilizer of x and by $\Gamma.x$ the orbit of x. The set of points of X fixed by Γ is denoted by X^Γ. If S is a subgroup of Γ, we denote by $N_\Gamma(S)$ the normalizer of S in Γ and by Γ^S the centralizer of S in Γ.

2.2 — Let us recall that G is a connected complex reductive group, $\mathcal{B}$ its flag variety and H a closed subgroup of G. We assume that H is *spherical*; that is, H has a dense orbit in $\mathcal{B}$. In this article, we are interested in the set $\mathbf{H}(\mathcal{B})$ of the orbits of H in $\mathcal{B}$. It is shown in [Bri86], [Vin86] or [Kno95] that $\mathbf{H}(\mathcal{B})$ is finite.

We recall the definition of [Res04] of a graph $\Gamma(G/H)$ whose vertices are the elements of $\mathbf{H}(\mathcal{B})$. The original construction of $\Gamma(G/H)$ due to M. Brion is very slightly different (see [Bri01]).

Consider the set Δ of conjugacy classes of minimal non solvable parabolic subgroups of G. If α belongs to Δ, we denote by $\mathcal{P}_\alpha$ the G-homogeneous space with isotropy α. Then, there exists a unique G-equivariant map $\phi_\alpha : \mathcal{B} \longrightarrow \mathcal{P}_\alpha$ which is a $\mathbb{P}^1$-bundle.

Let $V \in \mathbf{H}(\mathcal{B})$ and $\alpha \in \Delta$. We assume that the restriction of ϕ_α to V is finite and we denote its degree by $d(V, \alpha)$. Then, there exists a unique open H-orbit V' in $\phi_\alpha^{-1}(\phi_\alpha(V))$; in this case, we say that α *raises* V *to* V'. One of the following three cases occurs.

- Type U: H has two orbits in $\phi_\alpha^{-1}(\phi_\alpha(V))$ (V and V') and $d(V, \alpha) = 1$.
- Type T: H has three orbits in $\phi_\alpha^{-1}(\phi_\alpha(V))$ and $d(V, \alpha) = 1$.
- Type N: H has two orbits in $\phi_\alpha^{-1}(\phi_\alpha(V))$ (V and V') and $d(V, \alpha) = 2$.

Definition. Let $\Gamma(G/H)$ be the oriented graph with vertices the elements of $\mathbf{H}(\mathcal{B})$ and edges labeled by Δ, where V is joined to V' by an edge labeled by α if α raises V to V'. This edge is simple (resp. double) if $d(V, \alpha) = 1$ (resp. 2). Following the above cases, we say that an edge has *type* U*,* T *or* N.

One can find examples of graphs $\Gamma(G/H)$ in [Bri01, Pin01, Res04].

2.3 — Let us fix a Borel subgroup B of G, and a maximal torus T of B. Let W denote the Weyl group of T. We now describe Knop's action of W on the set $\mathbf{H}(\mathcal{B})$ (see also [Kno95]). Indeed, the action of simple reflexions easily reads off the graph $\Gamma(G/H)$.

Every α in Δ has a unique representative P_α which contains B. Moreover, there exists a unique s_α in W such that $Bs_\alpha B$ is dense in P_α; and this s_α is a simple reflexion of W. The map, $\Delta \longrightarrow W$, $\alpha \longmapsto s_\alpha$ is a bijection from Δ onto the set of simple reflexions of W.

Consider the group $\widetilde{W}$ generated by $\{s_\alpha \ : \ \alpha \in \Delta\}$ with the relations $s_\alpha^2 = 1$. There is a surjective homomorphism $\widetilde{W} \longrightarrow W$. Let $\mathcal{T}$ denote its kernel.

One defines an action of $\widetilde{W}$ on the set $\mathbf{H}(\mathcal{B})$ by describing the action of the s_α, for any $\alpha \in \Delta$:

- Type U: s_α exchanges the two vertices of an edge of type U labeled by α.
- Type T: If α raises V_1 and V_2 on V, then $s_\alpha V_1 = V_2$ and $s_\alpha V = V$.
- Type N: s_α fixes the two vertices of a double edge labeled by α.
- s_α fixes all others vertices of $\Gamma(G/H)$.

In [Kno95], F. Knop showed that this action of $\widetilde{W}$ factors through W; that is , that $\mathcal{T}$ acts trivially on $\mathbf{H}(\mathcal{B})$. The aim of this paper is to describe the orbits of this action by a natural invariant and to give some consequences.

2.4 — Denote by $\mathcal{H}$ the G-homogeneous space G/H. If V belongs to $\mathbf{H}(\mathcal{B})$, we set:

$$V_{\mathcal{H}} = \{gH/H \,:\, g^{-1}B/B \in V\}.$$

Then, $V_{\mathcal{H}}$ is a B-orbit in $\mathcal{H}$. Moreover, the map $V \longmapsto V_{\mathcal{H}}$ is a bijection from $\mathbf{H}(\mathcal{B})$ onto the set $\mathbf{B}(\mathcal{H})$ of B-orbits in $\mathcal{H}$.

The *character group* $\mathcal{X}(V_{\mathcal{H}})$ of V (or $V_{\mathcal{H}}$) is the set of all characters of B that arise as weights of eigenvectors of B in the function field $\mathbb{C}(V_{\mathcal{H}})$. Then $\mathcal{X}(V_{\mathcal{H}})$ is a free abelian group of finite rank $\mathrm{rk}(V_{\mathcal{H}})$ (or $\mathrm{rk}(V)$), *the rank of* V.

3 The type of an orbit of H

3.1 — In this section, we define the type of a H-orbit in general (not only in $\mathcal{B}$). We start with two technical lemmas.

Let us fix a maximal torus T^H of H. If V is a H-homogeneous space, we set:

$$\rho_V = \min_{x\in V} \dim(T^H.x).$$

Lemma 3.1 *Let $V \in \mathbf{H}(\mathcal{B})$. Then, for all $x \in V$, the following are equivalent:*

(i) $\dim(T^H.x) = \rho_V$,

(ii) $({T^H}_x)^\circ$ *is a maximal torus of* H_x.

Proof: Assume that $\dim(T^H.x) = \rho_V$. Let $S' \supseteq ({T^H}_x)^\circ$ be a maximal torus of H_x. Then, there exists h in H such that $h^{-1}S'h$ is contained is T^H. But, $h^{-1}S'h$ fixes $h^{-1}x$. Therefore, $\dim T^H - \dim T^H_x = \rho_V \le \dim(T^H.h^{-1}x) \le \dim T^H - \dim S'$; hence $\dim S' \le \dim {T^H}_x$. It follows that $S' = ({T^H}_x)^\circ$.

The converse is obvious since $({T^H}_x)^\circ$ is always a torus of H_x. $\square$

Lemma 3.2 *Let x and y belong to V such that* $\dim(T^H.x) = \dim(T^H.y) = \rho_V$. *Set* $S_x = ({T^H}_x)^\circ$ *and* $S_y = ({T^H}_y)^\circ$.

Then, we have:

(i) There exists h in H such that $y = h.x$ and $S_y = hS_xh^{-1}$.

(ii) There exist $\hat{w} \in N_H(T^H)$ such that $\hat{w}^{-1}S_y\hat{w} = S_x$ and $\hat{w}^{-1}.y \in H^{S_x}.x$.

Proof: Let $h_1 \in H$ such that $y = h_1.x$. By Lemma 3.1, $h_1^{-1}S_yh_1$ and S_x are maximal tori of $H_x = h_1^{-1}H_yh_1$. Therefore, (see [Hum75, 21.3]) there exists h_2 in H_x such that $h_2^{-1}h_1^{-1}S_yh_1h_2 = S_x$. Then, $h = h_1h_2$ satisfies Assertion 1.

Notice that $H^{S_x} = h^{-1}H^{S_y}h$. Then, T^H and $h^{-1}T^Hh$ are maximal tori of H^{S_x}; so there exists g_1 in H^{S_x} such that $g_1^{-1}h^{-1}T^Hhg_1 = T^H$. But, we have: $g_1^{-1}h^{-1}S_yhg_1 = S_x$. Then, $\hat{w} = hg_1$ satisfies Assertion 2. □

Let $W_H = N_H(T^H)/T^H$ denote the Weyl group of H. The group W_H acts by conjugacy on the set of subtori of T^H. Let V be a H-homogeneous space. Let us fix x in V such that $\rho_V = \dim(T^H.x)$. Then, by Lemma 3.2, the orbit $W_H.(T^H_x)^\circ$ does not depend on x but only on V; we call it *the type of V*.

3.2 — We have:

Proposition 3.1 *Let S belong to the type of V. Then, we have:*

(i) V^S is a unique orbit of $N_H(S)$.

(ii) The irreducible components of V^S are orbits of $(H^S)^\circ$.

Proof: Since V is stable by H, V^S is stable by $N_H(S)$. Let x and y belong to V^S. Let $h \in H$ such that $y = h.x$. Then, $h^{-1}Sh$ is contained in H_x. So by Lemma 3.1, S and $h^{-1}Sh$ are maximal tori of H_x and hence there exists h_1 in H_x such that $h_1^{-1}h^{-1}Shh_1 = S$. Then, $y = hh_1.x$ belongs to $N_H(S).x$. Assertion 1 is proved.

By [Hum75, Corollary 16.3], the identity component of $N_H(S)$ is $(H^S)^\circ$. Then, Assertion 2 follows from Assertion 1. □

4 The type of an orbit of H in $\mathcal{B}$

4.1 — In the previous section, we associated to each H-homogeneous space V a type and an integer ρ_V. Now, we apply these constructions to the orbits V of H in $\mathcal{B}$. First, Proposition 4.1 below shows that the type of V corresponds to the character group of V. We will deduce that $\rho_V - \mathrm{rk}(V)$ is independent of V.

Let us fix a maximal torus T of G containing T^H. Let B be a Borel subgroup of G containing T.

Proposition 4.1 *Let V be in $\mathbf{H}(\mathcal{B})$ and S be a subtorus of T which belongs to the type of V. Let $w \in W$ such that V intersects the irreducible component $G^S.wB/B$ of $\mathcal{B}^S$.*

Then, $\mathcal{X}(V) \otimes \mathbb{Q}$ is equal to $\mathcal{X}(T)^{w^{-1}Sw} \otimes \mathbb{Q}$.

Proof: Let $g \in G$ such that gB/B belongs to $V \cap G^S.wB/B$. Consider $y = g^{-1}H/H$. By replacing g by an element of gB, we may assume that $\dim(T.y) = \min_{y' \in B.y} \dim(T.y')$. But, by Lemma 3.1 T_y° is a maximal torus of B_y. Since the unipotent radical of B_y° is contained in U, it is equal to U_y. Then, we have: $G_y^\circ = T_y^\circ U_y$.

We have: $\mathcal{X}(V) \otimes \mathbb{Q} = \mathcal{X}(B)^{B_y^\circ} \otimes \mathbb{Q}$. Moreover, the restriction map from $\mathcal{X}(B_y^\circ)$ to $\mathcal{X}(T_y^\circ)$ is injective. Therefore, $\mathcal{X}(V) \otimes \mathbb{Q} = \mathcal{X}(T)^{T_y^\circ} \otimes \mathbb{Q}$.

Since $B_y = g^{-1}H_x g$, $g^{-1}Sg$ is a maximal torus of B_y. Therefore, there exists $b \in B_y^\circ$ such that $S = gbT_y^\circ b^{-1}g^{-1}$. By replacing g by gb (and keeping x and y unchanged), we may assume that b is trivial; that is, that $S = gT_y^\circ g^{-1}$.

It follows that T and gTg^{-1} are maximal tori of G^S. Then, there exists $s \in G^S$ such that sg normalizes T. Let w_1 be the class of sg in the Weyl group of T. Then, $T_y^\circ = w_1^{-1}Sw_1$.

On the other hand, since $sg \in G^S wB$, there exists w' in the Weyl group of G^S such that $w_1 = w'.w$. Then, $T_y^\circ = w^{-1}Sw$ and the proposition follows. □

Corollary 4.1 *Let V be an orbit of H in $\mathcal{B}$. We have:*

(i) $\rho_V - \mathrm{rk}(V) = \mathrm{rk}(G) - \mathrm{rk}(H)$.

(ii) The rank of V is minimal in $\mathbf{H}(\mathcal{B})$ if and only if V contains points fixed by T^H.

Proof: The proposition shows that the rank of V is the dimension of T minus the dimension of S. On the other hand, ρ_V is the difference between the rank of H and the dimension of S. Assertion 1 follows.

Since T^H has fixed points in $\mathcal{B}$, the rank of V is minimal if and only if $\rho_V = 0$; that is, if and only if V contains points fixed by T^H. □

4.2 — Let V be in $\mathbf{H}(\mathcal{B})$ and S belong to the type of V. We are now interested in the set V^S. We can make Proposition 3.1 more precise:

Proposition 4.2 *(i) The intersection of V^S and an irreducible component of $\mathcal{B}^S$ is a unique orbit of H^S.*

(ii) If H is connected, the intersection of V and one irreducible component of $\mathcal{B}^S$ is irreducible.

Proof: Let x and y be two points of V^S in the same irreducible component of $\mathcal{B}^S$. Since the irreducible components of $\mathcal{B}^S$ are orbits of G^S, there exists $g \in G^S$ such that $y = g.x$. By Assertion (i) there exists $h \in N_H(S)$ such that $y = h.x$. Then, $g^{-1}h$ belongs to G_x which is a Borel subgroup of G which contains S. Moreover, $g^{-1}h$ normalizes S. But, by [Hum75, Proposition 19.4], we have: $N_{G_x}(S) = G_x^S$. So, $g^{-1}h$ and h centralize S. Assertion (iii) follows.

If H is connected, Theorem 22.3 of [Hum75] shows that H^S is connected. Now, Assertion (iv) follows from Assertion (iii). $\square$

4.3 — We are now interested in the set of irreducible components of $\mathcal{B}^S$ which intersect V. By Proposition 4.2, if H is connected, this set is in bijection with the set of the irreducible components of V^S.

Since the irreducible components of $\mathcal{B}^S$ are the $G^S wB/B$ for w in W, we set:

$$\mathcal{C}(V, S) = \{w \in W \,:\, V \cap G^S wB/B \neq \emptyset\}.$$

To describe $\mathcal{C}(V, S)$, we need two technical lemmas.

Lemma 4.1 *Set* $N_H(S)G^S = \{hg \,:\, h \in N_H(S)$ and $g \in G^S\}$.

Then, $N_H(S)G^S$ *is a closed subgroup of* $N_G(S)$ *whose identity component is* G^S*. Moreover, the group* $(N_H(S)G^S)/G^S$ *is isomorphic to* $N_H(S)/H^S$ *(the Weyl group of* S *in* H*, denoted by* $W(H, S)$*).*

Proof: Notice that, $N_H(S)$ normalizes G^S. Now, one easily checks that $N_H(S)G^S$ is a subgroup of G. Moreover, $N_H(S)G^S$ is clearly contained in $N_G(S)$ and contains G^S. But by [Hum75, Corollary 16.3], G^S is the identity component of $N_G(S)$. It follows that the index of G^S in $N_H(S)G^S$ is finite. Then, $N_H(S)G^S$ is closed in $N_G(S)$ and its identity component is G^S. The last assertion is obvious. $\square$

Notice that T is contained in $N_H(S)G^S$. Set $W_{N_H(S)G^S} = N_{N_H(S)G^S}(T)/T$. Then, the inclusion of $N_{N_H(S)G^S}(T)$ in $N_G(T)$ induces an embedding of $W_{N_H(S)G^S}$ in W. Let W_{G^S} denote the Weyl group of (G^S, T).

Lemma 4.2 *We have an exact sequence:*

$$1 \longrightarrow W_{G^S} \longrightarrow W_{N_H(S)G^S} \longrightarrow W(H, S) \longrightarrow 1.$$

Proof: Let us start with the exact sequence given by Lemma 4.1:

$$1 \longrightarrow G^S \longrightarrow N_H(S)G^S \longrightarrow W(H, S) \longrightarrow 1.$$

By intersecting with $N_{N_H(S)G^S}(T)$, we obtain an exact sequence:

$$1 \longrightarrow N_{G^S}(T) \longrightarrow N_{N_H(S)G^S}(T) \longrightarrow W(H, S),$$

and it is sufficient to prove that the last map is surjective. Let h in $N_H(S)$ and g in G^S. Since, $ghT(gh)^{-1}$ and T are maximal tori of G^S, there exists $g' \in G^S$ such that $g'ghT(gh)^{-1}g'^{-1} = T$. The lemma follows. $\square$

If E is a finite set, let $|E|$ denote its cardinality. Now, we can describe $\mathcal{C}(V, S)$:

Proposition 4.3 *(i) The set* $\mathcal{C}(V, S)$ *is an orbit of* $W_{N_H(S)G^S}$ *for its action on* W *by left multiplication.*

(ii) If H is connected, V^S has $|W_{N_H(S)G^S}|$ irreducible components.

Proof: Let σ be an element of $\mathcal{C}(V,S)$ and let x belong to $V \cap G^S\sigma B/B$. By Proposition 3.1, $V^S = N_H(S).x$. Therefore $G^S.V^S = G^S N_H(S).x = (N_H(S)G^S)\sigma B/B$. But $G^S V^S$ is the union of the $G^S.wB/B$ for $w \in \mathcal{C}(V,S)$. The first assertion follows.

By Proposition 4.2, each irreducible component of V^S is the intersection of V and one irreducible component of $\mathcal{B}^S = \coprod_{w\in W_{G^S}\backslash W} G^S wB/B$. Therefore, by the first assertion V^S has $\frac{|W_{N_H(S)G^S}|}{|W_{G^S}|}$ irreducible components. Now, the second assertion follows from Lemma 4.2. $\square$

4.4 — Each irreducible component of $\mathcal{B}^S$ is isomorphic to the flag variety $\mathcal{B}_{G^S}$ of G^S. Moreover, by Proposition 4.2, V intersects any such irreducible component in one orbit of H^S. We will now describe the orbits of H^S in $\mathcal{B}_G$ which appear in that way.

Let τ be a W_H-orbit of subtori of T^H. Let $\mathbf{H}(\mathcal{B})_\tau$ denote the set of H-orbits in $\mathcal{B}$ of type τ.

Proposition 4.4 *Assume that $\mathbf{H}(\mathcal{B})_\tau$ is not empty. Let us fix an element S in τ. Then,*

(i) The subgroup H^S of G^S is spherical.

(ii) The rank of G^S/H^S is equal to the rank of the free abelian group $\mathcal{X}(T)^S$.

(iii) Let $V \in \mathbf{H}(\mathcal{B})_\tau$ and $x \in V^S$. Then, $\rho_{H^S.x} = \mathrm{rk}(H) - \mathrm{rk}(S)$. In particular, $\mathrm{rk}(H^S.x) = \mathrm{rk}(G^S/H^S)$.

(iv) Conversely, let y in $\mathcal{B}^S$ such that $\rho_{H^S.y} = \mathrm{rk}(H) - \mathrm{rk}(S)$. Then, the type of $H.y$ is τ.

Proof: We first prove Assertions 3 and 4. Let $V \in \mathbf{H}(\mathcal{B})_\tau$ and $x \in V^S$.

Let $y \in H^S.x$. Since y belongs to V and the type of V is τ, we have $\dim(T^{H}{}_y) \le \dim S$. Then, $\rho_{H^S.y} \le \mathrm{rk}(H) - \mathrm{rk}(S)$.

But $\rho_{H^S.x} \ge \rho_{H.x} = \mathrm{rk}(H) - \mathrm{rk}(S)$. So $\rho_{H^S.x} = \mathrm{rk}(H) - \mathrm{rk}(S)$. This proves Assertion 3.

Set $\Omega = \{y \in G^S.x \ : \ \rho_{H^S.y} \le \mathrm{rk}(H) - \mathrm{rk}(S)\}$. The set Ω is open in $G^S.x$ and contains x.

Let $y \in \Omega$. Then, S is a maximal torus of H^S_y. Let S_y be a maximal torus of H_y containing S. Then, S_y is contained in H^S. Therefore $S = S_y$. Then, Lemma 3.1 shows that $\rho_{H.y} = \mathrm{rk}(H) - \mathrm{rk}(S)$. Therefore, since $(H.y)^S$ is not empty, the type of $H.y$ is τ. By Corollary 4.1, this proves Assertion 4.

By Proposition 4.2, each orbit of type τ intersects $G^S.x$ in a unique orbit of H^S. Hence, Assertion 3 shows that the set of H^S-orbit in Ω is finite. So, H^S has a dense orbit in Ω and in $G^S.x$. The first assertion follows. The second one is now a consequence of Assertion 3. $\square$

5 Knop's action of W on $\mathbf{H}(\mathcal{B})$ and orbit type

5.1 — Keep the notation as above. In particular, τ is a W_H-conjugacy class of subtori of T^H such that $\mathbf{H}(\mathcal{B})_\tau$ is not empty and S belongs to τ. Set ${}_{W_{N_H(S)G^S}}\backslash W = \{W_{N_H(S)G^S}w : w \in W\}$. By Proposition 4.3, we can define a map

$$\begin{array}{rccl} \Theta : & \mathbf{H}(\mathcal{B})_\tau & \longrightarrow & {}_{W_{N_H(S)G^S}}\backslash W \\ & V & \longmapsto & \mathcal{C}(V, S). \end{array}$$

We consider on ${}_{W_{N_H(S)G^S}}\backslash W$ the action of the Weyl group W by right multiplication.

In this section we show the following

Theorem 1 *The subset $\mathbf{H}(\mathcal{B})_\tau$ of $\mathbf{H}(\mathcal{B})$ is stable by Knop's action of W. Moreover, the map Θ is W-equivariant.*

5.2 — Start with

Lemma 5.1 *Let $V \in \mathbf{H}(\mathcal{B})_\tau$, $x \in V^S$ and $\alpha \in \Delta$. Consider $\phi_\alpha : \mathcal{B} \longrightarrow \mathcal{P}_\alpha$. Let $w \in W$ be such that $G^S.x = G^S.wB/B$. Then one of the two following cases occurs:*

Case 1: $\phi_\alpha^{-1}(\phi_\alpha(x))$ is pointwise fixed by S.
Then, we have $G^S ws_\alpha B/B = G^S wB/B$.

Case 2: There exists $y \neq x$ such that $\phi_\alpha^{-1}(\phi_\alpha(x))^S = \{x, y\}$.
Then, $G^S.x \neq G^S.y$ and $G^S.y = G^S ws_\alpha B/B$.

Proof: Set $F = \phi_\alpha^{-1}(\phi_\alpha(x))$. The variety F is isomorphic to the projective line $\mathbb{P}^1$. Moreover, F is stable by the action of the torus S. Then, the image of S in $\mathrm{Aut}(F) \simeq PSL(2)$ is either trivial or a maximal torus of $\mathrm{Aut}(F)$. In particular, one of the following cases occurs.

Case 1: $F^S = F$.

Case 2: There exists $y \neq x$ such that $F^S = \{x, y\}$.

In either case, consider the G^S-orbit $G^S.\phi_\alpha(x)$ and the flag variety $\mathcal{B}_{G^S}$ of the group G^S. Since $G^S.\phi_\alpha(x)$ is the image by ϕ_α of $G^S.x \simeq \mathcal{B}_{G^S}$, it is a complete G^S-homogeneous space. Moreover, since ϕ_α is a $\mathbb{P}^1$-fibration, we have: $\dim(\mathcal{B}_{G^S}) \geq \dim(G^S.\phi_\alpha(x)) \geq \dim(\mathcal{B}_{G^S}) - 1$. Then, two cases occur.

Case a: $G^S_{\phi_\alpha(x)}$ is a non solvable minimal parabolic subgroup of G^S and $G^S.x$ contains F.

Case b: $G^S_{\phi_\alpha(x)}$ is a Borel subgroup of G^S and $F \cap G^S.x = \{x\}$.

In Case 1, F is contained in the irreducible component of $\mathcal{B}^S$ which contains x; that is in $G^S.x$. So, Case 1 implies Case a. In Case 2, we cannot have that F contains $G^S.x$. So, Case 2 implies Case b. In particular, $G^S.x \neq G^S.y$.

It remains to determine $G^S.ws_\alpha B/B$ in each case. The fiber $\phi_\alpha^{-1}(\phi_\alpha(B/B))$ of ϕ_α is the closure $\overline{Bs_\alpha B}/B$ of $Bs_\alpha B/B$ in $\mathcal{B}$. Let $g \in G^S$ be such that $x = gwB/B$. Then, $F = gw\overline{Bs_\alpha B}/B$.

In Case 1, F is contained in $G^S wB/B$. In particular, gws_α belongs to $G^S wB/B$. Therefore, $G^S ws_\alpha B/B = G^S wB/B$.

In Case 2, we can notice that $gws_\alpha B/B$ is fixed by S and belongs to F; Therefore, $y = gws_\alpha B/B$. Then, $G^S.y = G^S ws_\alpha B/B$. $\square$

5.3 — Proof of Theorem 1. Let $V \in \mathbf{H}(\mathcal{B})_\tau$ and $\alpha \in \Delta$. We will prove that $\mathcal{C}(V,S)s_\alpha = \mathcal{C}(s_\alpha V, S)$. Let $w \in \mathcal{C}(V,S)$. By Proposition 4.3, it is sufficient to show that ws_α belongs to $\mathcal{C}(s_\alpha V, S)$.

We fix x in $V^S \cap G^S wB/B$ and we set $F = \phi_\alpha^{-1}(\phi_\alpha(x))$. Then, one of the following 4 cases occurs.

<u>Case 1:</u> α raises V on $s_\alpha V$ (type U).

Since $V \cap F = \{x\}$, $(s_\alpha V)^S$ is not empty. Since V and $s_\alpha V$ have the same rank, Corollary 4.1 implies that S belongs to the type of $s_\alpha V$.

Let us assume that there exists $y \neq x$ such that $F^S = \{x, y\}$. Necessarily, y belongs to $s_\alpha V$. But Lemma 5.1 shows that $G^S.y = G^S ws_\alpha B/B$. So, ws_α belongs to $\mathcal{C}(s_\alpha V, S)$.

If $F^S = F$ then Lemma 5.1 shows that F is contained in $G^S wB/B = G^S ws_\alpha B/B$. Then, since F intersects $s_\alpha V$, ws_α belongs to $\mathcal{C}(s_\alpha V, S)$.

<u>Case 2:</u> α raises V and $s_\alpha V$ on a third H-orbit V_1 (type T).

Since $\mathrm{rk}(V_1) = \mathrm{rk}(V) + 1$, Corollary 4.1 shows that V_1^S is empty. Then, by Lemma 5.1 there exists $y \neq x$ such that $F^S = \{x, y\}$. On the other hand, $F \cap V_1$ is equal to F with two points removed (type T). Since, V_1^S is empty it follows that $F \cap V_1 = F - \{x, y\}$, $F \cap V = \{x\}$ and $F \cap s_\alpha V = \{y\}$. But, Lemma 5.1 shows that $G^S.y = G^S ws_\alpha B/B$. Therefore, ws_α belongs to $\mathcal{C}(s_\alpha V, S)$.

<u>Case 3:</u> α raises V and $s_\alpha V = V$ (type N).

The same proof as in Case 2 shows that $F^S = \{x, y\} = F \cap V$ and $G^S.y = G^S ws_\alpha B/B$. It follows that ws_α belongs to $\mathcal{C}(V,S) = \mathcal{C}(s_\alpha V, S)$.

<u>Case 4:</u> $F \cap V$ is open in F.

If $F^S = F$ then V is the only H-orbit in $\phi_\alpha^{-1}(V)$ of maximal rank (type T or N). Therefore, $s_\alpha V = V$. Moreover, by Lemma 5.1, we have $G^S.wB/B = G^S ws_\alpha B/B$; therefore, $ws_\alpha \in \mathcal{C}(V,S) = \mathcal{C}(s_\alpha V, S)$.

We may assume that there exists $y \neq x$ such that $F^S = \{x, y\}$. Then, since $F \cap V$ is open in F, stable by S and contains x, $F \cap V$ is either F or $F - \{y\}$. If $F \cap V = F - \{y\}$ then α raises $s_\alpha V$ to V by an edge of type U. By exchanging V and $s_\alpha V$ we come back to Case 1. Assume that V contains F. Since $G^S ws_\alpha B/B$ intersects F, it intersects V. Then, $V = s_\alpha V$ and $ws_\alpha \in \mathcal{C}(V,S) = \mathcal{C}(s_\alpha V, S)$.

This completes the proof of Theorem 1. $\square$

5.4 — Let σ be in W and $\overline{\sigma}$ be its class in ${}_{W_{N_H(S)G^S}}\backslash W$. We are now interested in the fiber $\Theta^{-1}(\overline{\sigma})$ of Θ. By definition of $\mathcal{C}(V,S)$, $\Theta^{-1}(\overline{\sigma})$ is the set of the orbits V in $\mathbf{H}(\mathcal{B})_\tau$ which intersects $G^S\sigma B/B$. Let $\mathbf{H}^S(\mathcal{B}_{G^S})$ denote the set of the H^S-orbits in the flag manifold $\mathcal{B}_{G^S}$ of

G^S, and let $\mathbf{H}^S(\mathcal{B}_{G^S})_{\max}$ denote the set of the H^S-orbits of maximal rank. By Proposition 4.4, the map

$$\begin{array}{rccl} \eta_\sigma & : \Theta^{-1}(\overline{\sigma}) & \longrightarrow & \mathbf{H}^S(\mathcal{B}_{G^S})_{\max} \\ & V & \longmapsto & V \cap G^S \sigma B/B \end{array}$$

is a bijection.

The subgroup $\sigma^{-1} W_{N_H(S)G^S} \sigma$ stabilizes $\Theta^{-1}(\overline{\sigma})$. Moreover, $W_{N_H(S)G^S}$ contains W_{G^S}. Therefore, the group W_{G^S} acts on $\Theta^{-1}(\overline{\sigma})$ through the morphism $W_{G^S} \longrightarrow W$, $w \longmapsto \sigma^{-1} w \sigma$. On the other hand, W_{G^S} acts on $\mathbf{H}^S(\mathcal{B}_{G^S})_{\max}$ by Knop's action. Is the bijection η_σ W_{G^S}-equivariant ? The answer is NO in general, but YES for at least one σ.

Proposition 5.1 *There exists σ such that η_σ is W_{G^S}-equivariant.*

Proof: Actually, the map Θ depends on the choice of the Borel subgroup B made in Paragraph 1. To prove the proposition, it is sufficient to prove that for a good choice of B, η_1 is W_{G^S}-equivariant. Let us make such a choice.

Let P be a parabolic subgroup of G with Levi subgroup G^S. Let B be a Borel subgroup of G such that $T \subset B \subset P$.

Notice that $B^S = B \cap G^S$ is a Borel subgroup of G^S. Denote by Δ^S the set of conjugacy classes of minimal non solvable parabolic subgroups of G^S. Let $\alpha \in \Delta^S$ and $\mathcal{P}^S_\alpha$ denote the G^S-homogeneous space with isotropy α. If P^S_α is a minimal parabolic subgroup of G^S containing B^S corresponding to α, then $P^S_\alpha.B$ is a minimal parabolic subgroup of G. Moreover, $P^S_\alpha = (P^S_\alpha.B) \cap G^S$. Therefore, we obtain an immersion (from now on implicit) of Δ^S in Δ. In particular $P_\alpha = P^S_\alpha B$. Consider the following commutative diagram $\mathcal{D}$:

$$\begin{tikzcd} \mathcal{B}_{G^S} \simeq G^S B/B \arrow[r, hook, "\text{inclusion}"] \arrow[d] & \mathcal{B} \arrow[d, "\phi_{\alpha_i}"] \\ \mathcal{P}^S_\alpha \simeq \mathcal{P}^S_{\alpha_i} \arrow[r, hook] & \mathcal{P}_{\alpha_i} \end{tikzcd}$$

The restriction of ϕ_α to $G^S B/B$ is obviously the unique G^S-equivariant map $\phi_{\alpha,S}$ from $\mathcal{B}_{G^S}$ onto $\mathcal{P}_{\alpha,S}$.

Let $x \in G^S B/B$ such that $H^S.x$ belongs to $\mathbf{H}^S(\mathcal{B}_{G^S})_{\max}$. It remains to prove the following

<u>Claim</u>: $G^S B/B \cap (s_\alpha.Hx) = s_\alpha(H^S x)$.

Since Diagram $\mathcal{D}$ is commutative, we have

$$\phi_\alpha^{-1}(\phi_\alpha(x)) = \phi_{\alpha,S}^{-1}(\phi_{\alpha,S}(x)); \tag{1}$$

we denote by F this subvarity of $\mathcal{B}$. Moreover, since the rank of $H^S x$ is maximal in $\mathbf{H}^S(\mathcal{B}_{G^S})$, Proposition 4.4 shows

$$G^S B/B \cap Hx = H^S x. \qquad (2)$$

Four cases can occur:
Case 1: α raises $H^S x$ in $\Gamma(G^S/H^S)$.
Case 2: α raises an orbit $H^S y$ of $\mathbf{H}^S(\mathcal{B}_{G^S})_{\max}$ on $H^S x$.
Case 3: α raises an orbit of $\mathbf{H}^S(\mathcal{B}_{G^S})$ on $H^S x$ by an edge of type T or N.
Case 4: $H^S x = \phi_{\alpha,S}^{-1}(\phi_{\alpha,S}(H^S x))$.

In Case 1, $F \cap H^S x = \{x\}$ and H^S_x, and hence H_x, acts transitively on $F - \{x\}$. Moreover, by Equality 2, $F \cap Hx = \{x\}$. Therefore, α raises Hx by an edge of type U in $\Gamma(G/H)$. The claim follows.

In Case 2, $H^S y$ is in Case 1. The claim follows.

In Case 3, $F \cap H^S x = F \cap Hx$ is equal to F with two points removed. Therefore, α raises an orbit of $\mathbf{H}(\mathcal{B})$ on Hx by an edge of type T or N, and $s_\alpha.Hx = Hx$.

In Case 4, F is contained in $H^S x$ and hence in Hx. As a consequence, $Hx = \phi_\alpha^{-1}(\phi_\alpha(Hx))$ and $s_\alpha.Hx = Hx$. This completes the proof of the proposition. □

Here, comes our main result.

Theorem 2 *Two elements of $\mathbf{H}(\mathcal{B})$ are in the same W-orbit for Knop's action if and only if they have the same type.*

Proof: By Theorem 1, it is sufficient to prove that one (or any) fiber of Θ is an orbit of $W_{N_H(S)G^S}$. Then, by Proposition 5.1, it is sufficient to prove the theorem for the orbits of maximal rank. Let V_0 be such an orbit. There exist a sequence $\alpha_1, \cdots, \alpha_k$ in Δ and a sequence $V_0, V_1, \cdots, V_k$ of H-orbits such that α_i raises V_{i-1} on V_i for all $i = 1, \cdots, k$, and V_k is the open H-orbit in $\mathcal{B}$. Since the rank of V_0 is maximal, all the orbits V_i have the same rank and the edges joining these orbits are of type U. Therefore, we have $(s_{\alpha_k} \cdots s_{\alpha_1}).V_0 = V_k$. The theorem is proved. □

6 Some consequences

6.1 — Theorem 2 has a nice corollary about the character groups of the elements of $\mathbf{B}(\mathcal{H})$:

Corollary 6.1 *Let V and V' in $\mathbf{H}(\mathcal{B})$. Then, $\mathcal{X}(V) = \mathcal{X}(V')$ if and only if $\mathcal{X}(V) \otimes \mathbb{Q} = \mathcal{X}(V') \otimes \mathbb{Q}$.*

Proof: Let us fix $T \subset B$. By identifying $\mathcal{X}(B)$ with $\mathcal{X}(T)$, we obtain an action of W on $\mathcal{X}(B)$. Assume that $\mathcal{X}(V) \otimes \mathbb{Q} = \mathcal{X}(V') \otimes \mathbb{Q}$. By Proposition 4.1, the orbits V and V' have the same type. Then, by Theorem 2, there exists w in W such that $V = wV'$.

Then, by [Kno95, Theorem 4.3], $\mathcal{X}(V) = w.\mathcal{X}(V')$. Now, $\mathcal{X}(V)\otimes\mathbb{Q} = \mathcal{X}(V')\otimes\mathbb{Q}$ implies $\mathcal{X}(V) = \mathcal{X}(V')$. $\square$

6.2 — We can also apply Theorem 2 to the description of the isotropy subgroups of the action of H in $\mathcal{B}$.

Corollary 6.2 *Let x and y be in $\mathcal{B}$ such that Hx and Hy have the same type. Then, (H_x/H_x°) and (H_y/H_y°) are isomorphic.*

Proof: Set $V = Hx$ and $V' = Hy$. Let $\alpha \in Delta$. Since W is generated by the simple reflections, by Theorem 2 it is sufficient to prove the corollary for $V' = s_\alpha.V \neq V$. Two cases occur:

- Type T: V and V' are raised on a third orbit V''.
- Type U: α raises V on V' (up to re-indexing).

In the first case, the restrictions of ϕ_α to V and V' are isomorphisms onto $\phi_\alpha(V'')$. The corollary follows.

Assume that α raises V on $V' = s_\alpha.V$. By replacing y by another point of Hy, we may assume that $\phi_\alpha(x) = \phi_\alpha(y)$. Since the restriction of ϕ_α to V is an isomorphism onto $\phi_\alpha(V)$ and $\phi_\alpha(V') = \phi_\alpha(V)$, H_y is contained in H_x. This inclusion induces a morphism $\psi\,:\, H_y/H_y^\circ \longrightarrow H_x/H_x^\circ$. But, H_x/H_y is isomorphic to $\mathbb{A}^1$ and hence irreducible. We deduce that ψ is surjective.

It remains to show that $H_y\cap H_x^\circ = H_y^\circ$ to prove that ψ is injective. Obviously, $H_y^\circ \subset (H_y\cap H_x^\circ)$; and we can define a morphism $H_x^\circ/H_y^\circ \longrightarrow H_x^\circ/(H_y\cap H_x^\circ)$. Since $H_x^\circ/(H_y\cap H_x^\circ)$ is isomorphic to $\mathbb{A}^1$, it is simply connected and $H_y\cap H_x^\circ = H_y^\circ$. $\square$

6.3 — We are going to apply Theorem 2 to the H-orbits in $\mathcal{B}$ of minimal rank. We keep notation as above. In particular, $\mathbf{H}(\mathcal{B})_{\{T^H\}}$ is the set of the orbits of H in $\mathcal{B}$ of minimal rank.

Proposition 6.1 *We assume that H is connected. Then, we have:*

(i) The group H^{T^H}/T^H is a maximal unipotent subgroup of G^{T^H}/T^H.

(ii) The stabilizers in W (for Knop's action) of the elements of $\mathbf{H}(\mathcal{B})_{\{T^H\}}$ are isomorphic to the Weyl group W_H of H.

(iii) Let V be in $\mathbf{H}(\mathcal{B})_{\{T^H\}}$. The stabilizers in H of the points of V are connected.

Proof: Since T^H is maximal in H, H^{T^H}/T^H is unipotent. But it is a spherical subgroup of G^{T^H}/T^H. Assertion 1 follows.

We claim that the cardinality of the set $\mathbf{H}(\mathcal{B})_{\{T^H\}}$ is $\frac{|W|}{|W_H|}$. By Proposition 4.3, we have to prove that the set of irreducible components of the V^{T^H} for $V\in\mathbf{H}(\mathcal{B})_{\{T^H\}}$ has the same

cardinality as W. But, by Proposition 4.4, this set is in natural bijection with the set of orbits of H^{T^H} in $\mathcal{B}^{T^H}$. Moreover, by Assertion 1, H^{T^H} has $|W_{G^{T^H}}|$ orbits in each one of the $\frac{|W|}{|W_{G^{T^H}}|}$ irreducible components of $\mathcal{B}^{T^H}$. The claim follows.

By Proposition 5.1, we may assume that η_1 is $W_{G^{T^H}}$-equivariant to prove Assertion 2. Let V be in $\mathbf{H}(\mathcal{B})_{\{T^H\}}$ such that $\Theta(V) = \overline{1}$. We have to prove that the stabilizer W_V of V in W is isomorphic to W_H. Since Θ is W-equivariant, W_V is contained in $W_{N_H(T^H)G^{T^H}}$ and by Lemma 4.2 maps on W_H. Moreover, the claim shows that $|W_V| = |W_H|$. So, by Lemma 4.2 it is sufficient to prove that $W_V \cap W_{G^S}$ is trivial. By Proposition 5.1, this is a consequence of Assertion 1.

By Corollary 6.2, it is sufficient to prove the last assertion for a closed orbit V of H in $\mathcal{B}$. Let x be in V. Since V is closed in $\mathcal{B}$, it is projective. So, H_x is a parabolic subgroup of H. In particular, it is connected. □

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References

[Bri86] M. Brion – "Quelques propriétés des espaces homogènes sphériques", *Manuscripta Math.* **55** (1986), no. 2, p. 191–198.

[Bri01] — , "On orbit closures of spherical subgroups in flag varieties", *Comment. Math. Helv.* **76** (2001), no. 2, p. 263–299.

[Hum75] J. Humphreys – *Linear algebraic groups*, Springer Verlag, New York, 1975.

[Kno95] F. Knop – "On the set of orbits for a Borel subgroup", *Comment. Math. Helv.* **70** (1995), no. 2, p. 285–309.

[Pin01] S. Pin – "Sur les singularités des orbites d'un sous-groupe de Borel dans les espaces symétriques", *Thèse, Université Grenoble I* (2001), p. 1–109, `http://www-fourier.ujf-grenoble.fr/THESE/these_daterev.html`.

[Res04] N. Ressayre – "Sur les orbites d'un sous-groupe sphérique dans la variété des drapeaux", *à paraitre aux Bull. de la SMF* (2004), p. 1–26.

[Vin86] È. B. Vinberg – "Complexity of actions of reductive groups", *Funktsional. Anal. i Prilozhen.* **20** (1986), no. 1, p. 1–13, 96.

[Wol93] J. A. Wolf – "Admissible representations and geometry of flag manifolds", in *The Penrose transform and analytic cohomology in representation theory (South Hadley, MA, 1992)*, Amer. Math. Soc., Providence, RI, 1993, p. 21–45.

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